

A note on Veraverbeke's theorem

Stan Zachary

*Heriot-Watt University
Edinburgh*

We give an elementary probabilistic proof of Veraverbeke's Theorem for the asymptotic distribution of the maximum of a random walk with negative drift and heavy-tailed increments. The proof gives insight into the principle that the maximum is in general attained through a single large jump.

1 Introduction

Veraverbeke's Theorem (Veraverbeke (1977), Embrechts and Veraverbeke (1982)) gives the asymptotic distribution of the maximum M of a random walk with negative drift and heavy-tailed increments, and has well-known applications to both queueing theory and risk theory. A simpler treatment of this result is given, for example, by Embrechts *et al.* (1997) or Asmussen (2000), and is based on renewal theory and ascending ladder heights. Nevertheless the underlying intuition of the result is that the only significant way in which a high value of the maximum can be attained is through “one big jump” by the random walk away from its mean path. We give here a relatively short proof from first principles which captures this intuition. It is similar in spirit to the existing probabilistic proof, but by considering instead a first renewal time at which the random walk exceeds a “tilted” level, the argument becomes more elementary. In particular subsequent renewals have an asymptotically negligible probability under appropriate limits, and results from renewal theory—notably the derivation and use of the Pollaczek-Khinchine formula—are not required. We consider separately lower and upper bounds, as a slightly weaker condition is required for the former. Further, this treatment facilitates the derivation of bounds for $\mathbf{P}(M > x)$ for any x —see Remark 2 below. The argument leading to the lower bound is implicit in much of the literature (see, for example, Asmussen *et al.* (1999), Korshunov (2002), Baccelli and Foss (2003, Theorem 4)). We give it here for completeness. The argument leading to the upper bound is new.

2 Veraverbeke's Theorem.

We recall first some definitions and known properties. For any distribution function H on \mathbb{R} let $\overline{H}(x) = 1 - H(x)$ for all x . A distribution function H on \mathbb{R}_+ is *subexponential* if and only if $\overline{H}(x) > 0$ for all x and

$$\lim_{x \rightarrow \infty} \overline{H^{*n}}(x) / \overline{H}(x) = n, \quad n \geq 2, \quad (1)$$

where H^{*n} is the n -fold convolution of H with itself. (It is sufficient to verify the condition (1) in the case $n = 2$.) More generally, a distribution function H on \mathbb{R} is subexponential if and only if H^+ is subexponential, where $H^+ = H\mathbf{I}_{\mathbb{R}_+}$ and $\mathbf{I}_{\mathbb{R}_+}$ is the indicator

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function of \mathbb{R}_+ . In this case the condition (1) continues to hold, but is no longer sufficient for subexponentiality. It is well known that if H is subexponential, then H is *long-tailed*, i.e.

$$\lim_{x \rightarrow \infty} \frac{\overline{H}(x-h)}{\overline{H}(x)} = 1, \quad \text{for all fixed } h \in \mathbb{R}. \quad (2)$$

(See, for example, Embrechts *et al.* (1997, Lemma 1.3.5). It is of course sufficient to require that (2) hold for $h > 0$; the extension to $h < 0$ follows by taking reciprocals.)

For any distribution function H on \mathbb{R} with finite mean define also the integrated, or *second-tail*, distribution function H^s by

$$\overline{H^s}(x) = \min\left(1, \int_x^\infty \overline{H}(t) dt\right). \quad (3)$$

(When H is supported on \mathbb{R}_+ , the second-tail distribution is usually, and in this case more naturally, defined instead by $\overline{H^s}(x) = \mu^{-1} \int_x^\infty \overline{H}(t) dt$, where $\mu = \int_0^\infty \overline{H}(t) dt$ is the mean of H .)

Now let $\{\xi_n\}_{n \geq 1}$ be independent identically distributed random variables with distribution function F such that

$$\mathbf{E}\xi_1 = -a < 0. \quad (4)$$

Let $S_0 = 0$, $S_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$. Let $M_n = \max_{0 \leq i \leq n} S_i$ for $n \geq 0$ and let $M = \sup_{n \geq 0} S_n$. Clearly $\mathbf{P}(M < \infty) = 1$. We are interested in the asymptotic distribution of $\mathbf{P}(M > x)$ as $x \rightarrow \infty$. Under an appropriate further condition on the second-tail distribution function F^s this is given by Theorem 1 below. Part (ii) of this result is Veraverbeke's Theorem. The proof we give is also well-adapted to the possibility of obtaining bounds for $\mathbf{P}(M > x)$ for any x .

Theorem 1 (Veraverbeke).

(i) Suppose that, in addition to (4), F^s is long-tailed. Then

$$\liminf_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F^s}(x)} \geq \frac{1}{a}. \quad (5)$$

(ii) Suppose that, in addition to (4), F^s is subexponential. Then

$$\lim_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F^s}(x)} = \frac{1}{a}. \quad (6)$$

Proof. We derive first the lower bound given in (i). From (4), by the Weak Law of Large Numbers, given $\epsilon > 0$, $\delta > 0$, we can choose $L \equiv L_{\epsilon, \delta}$ such that

$$\mathbf{P}(S_n > -L - n(a + \epsilon)) \geq 1 - \delta, \quad n = 0, 1, 2, \dots \quad (7)$$

Then, for $x \geq 0$,

$$\begin{aligned} \mathbf{P}(M > x) &= \sum_{n \geq 0} \mathbf{P}(M_n \leq x, S_{n+1} > x) \\ &\geq \sum_{n \geq 0} \mathbf{P}(M_n \leq x, S_n > -L - n(a + \epsilon), \xi_{n+1} > x + L + n(a + \epsilon)) \\ &= \sum_{n \geq 0} \mathbf{P}(M_n \leq x, S_n > -L - n(a + \epsilon)) \mathbf{P}(\xi_{n+1} > x + L + n(a + \epsilon)) \\ &\geq \sum_{n \geq 0} (1 - \delta - \mathbf{P}(M_n > x)) \overline{F}(x + L + n(a + \epsilon)) \\ &\geq (1 - \delta - \mathbf{P}(M > x)) \frac{\overline{F^s}(x + L)}{a + \epsilon}, \end{aligned}$$

where the last line above follows since, from (3), for any x and any constant $b > 0$, $\overline{F^s}(x) \leq b \sum_{n \geq 0} \overline{F}(x + nb)$. Rearranging, we obtain

$$\mathbf{P}(M > x) \geq \frac{(1 - \delta)\overline{F^s}(x + L)}{a + \epsilon + \overline{F^s}(x + L)} \quad (8)$$

Since F^s is assumed long-tailed, it now follows that

$$\liminf_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F^s}(x)} \geq \frac{1 - \delta}{a + \epsilon}.$$

The result (5) is thus obtained on letting $\delta, \epsilon \rightarrow 0$.

We now prove (ii). Since F^s is here assumed subexponential, it is in particular long-tailed. Hence, by (i), it is sufficient to establish the upper bound associated with the result (6). For a sequence of events $\{A_n\}$ we make the convention: $\min\{n \geq 1 : \mathbf{I}(A_n) = 1\} = \infty$ if $\mathbf{I}(A_n) = 0$ for all n . Given $\epsilon \in (0, a)$ and $R > 0$, define renewal times $0 \equiv \tau_0 < \tau_1 \leq \tau_2 \leq \dots$ for the process $\{S_n\}$ by

$$\tau_1 = \min\{n \geq 1 : S_n > R - n(a - \epsilon)\} \leq \infty,$$

and, for $m \geq 2$,

$$\begin{aligned} \tau_m &= \infty, \quad \text{if } \tau_{m-1} = \infty, \\ \tau_m &= \tau_{m-1} + \min\{n \geq 1 : S_{\tau_{m-1}+n} - S_{\tau_{m-1}} > R - n(a - \epsilon)\}, \quad \text{if } \tau_{m-1} < \infty. \end{aligned}$$

Observe that

$$\{(\tau_m - \tau_{m-1}, S_{\tau_m} - S_{\tau_{m-1}})\}_{m \geq 1} \quad \text{are i.i.d.} \quad (9)$$

Then, again from (4), by the Strong Law of Large Numbers,

$$\gamma \equiv \mathbf{P}(\tau_1 < \infty) \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (10)$$

Define also $S_\infty = -\infty$. We now have that, for all sufficiently large x ,

$$\begin{aligned} \mathbf{P}(S_{\tau_1} > x) &= \sum_{n \geq 1} \mathbf{P}(\tau_1 = n, S_n > x) \\ &\leq \sum_{n \geq 1} \mathbf{P}(S_{n-1} \leq R - (n-1)(a - \epsilon), S_n > x) \\ &\leq \sum_{n \geq 1} \overline{F}(x - R + (n-1)(a - \epsilon)) \\ &\leq \frac{1}{a - \epsilon} \overline{F^s}(x - R - a + \epsilon). \end{aligned} \quad (11)$$

where the last line above follows since, from (3), for all sufficiently large x and any constant $b > 0$, $\overline{F^s}(x) \geq b \sum_{n \geq 1} \overline{F}(x + nb)$.

Let $\{\phi_m\}_{m \geq 1}$ be independent identically distributed random variables such that

$$\mathbf{P}(\phi_1 > x) = \mathbf{P}(S_{\tau_1} > x \mid \tau_1 < \infty), \quad x \in \mathbb{R}.$$

Then, from (11) and since F^s is long-tailed,

$$\mathbf{P}(\phi_1 > x) \leq \overline{G}(x), \quad x \in \mathbb{R}, \quad (12)$$

for some distribution function G on \mathbb{R} satisfying

$$\lim_{x \rightarrow \infty} \frac{\overline{G}(x)}{\overline{F^s}(x)} = \frac{1}{\gamma(a - \epsilon)}. \quad (13)$$

We now have, for all $x > R - a + \epsilon$,

$$\mathbf{P}(M > x) \leq \sum_{m \geq 1} \mathbf{P}(M > x, S_{\tau_m} > x - R + a - \epsilon) \quad (14)$$

$$\leq \sum_{m \geq 1} \gamma^m \mathbf{P}(\phi_1 + \cdots + \phi_m > x - R + a - \epsilon) \quad (15)$$

$$\leq \sum_{m \geq 1} \gamma^m \overline{G^{*m}}(x - R + a - \epsilon), \quad (16)$$

where (15) and (16) follow from (9) and (12) respectively.

It follows from (13), the subexponentiality of F^s , and the well-known properties of subexponential distributions (see, for example, Embrechts *et al.* (1997), Sigman (1999), or Asmussen (2000)) that G is subexponential. Hence from the well-known upper bound for convolutions of subexponential distributions (again see, for example, Embrechts *et al.* (1997, Lemma 1.3.5)), given any $k > 1$, there exists $A > 0$ such that

$$\frac{\overline{G^{*m}}(x)}{\overline{F^s}(x)} \leq Ak^m \quad \text{for all } x \text{ and for all } m \geq 0, \quad (17)$$

and also, again from (13) and by the subexponentiality of G , for all $m \geq 1$,

$$\lim_{x \rightarrow \infty} \frac{\overline{G^{*m}}(x)}{\overline{F^s}(x)} = \frac{m}{\gamma(a - \epsilon)}. \quad (18)$$

Hence by (16), (18), the long-tailedness of F^s , and the dominated convergence theorem (justified by (17) for any k such that $k\gamma < 1$),

$$\begin{aligned} \limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F^s}(x)} &\leq \frac{1}{a - \epsilon} \sum_{m \geq 1} m \gamma^{m-1} \\ &= \frac{1}{(a - \epsilon)(1 - \gamma)^2}. \end{aligned}$$

Now let $R \rightarrow \infty$, so that $\gamma \rightarrow 0$ by (10), and then let $\epsilon \rightarrow 0$ to obtain the required upper bound

$$\limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F^s}(x)} \leq \frac{1}{a},$$

and hence the result (6). \square

Remark 1. It follows from (14) and the subsequent argument that, given ϵ , R , τ_1 , γ as defined in the above proof of the upper bound,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F^s}(x)} &\leq \limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M > x, S_{\tau_1} > x - R + a - \epsilon)}{\overline{F^s}(x)} + \gamma \left(\frac{2 - \gamma}{(a - \epsilon)(1 - \gamma)^2} \right) \\ &= \limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M > x, S_{\tau_1} > x)}{\overline{F^s}(x)} + \gamma \left(\frac{2 - \gamma}{(a - \epsilon)(1 - \gamma)^2} \right) \end{aligned} \quad (19)$$

$$= (1 + o(1)) \limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M > x, S_{\tau_1} > x)}{\overline{F^s}(x)} \quad \text{as } R \rightarrow \infty. \quad (20)$$

Here (19) follows from the observation that, for any $b > 0$, $\mathbf{P}(M > x, S_{\tau_1} > x) \leq \mathbf{P}(M > x, S_{\tau_1} > x - b) \leq \mathbf{P}(M > x - b, S_{\tau_1} > x - b)$, together with the long-tailedness of F^s again, while (20) follows from (10). Thus, given *any* $\epsilon \in (0, a)$, if R is chosen sufficiently large, then as $x \rightarrow \infty$ the only significant way in which M can exceed x is that it should do so at the first time n such that S_n exceeds $R - n(a - \epsilon)$. This is the principle of “one big jump”.

Remark 2. The proof of the lower bound above may be incorporated into that of the upper bound. It is necessary to modify the definition of τ_1 above to

$$\tau_1 = \min\{n \geq 1 : S_n < -R - n(a + \epsilon) \quad \text{or} \quad S_n > R - n(a - \epsilon)\}$$

and to similarly modify the definition of the subsequent renewal times τ_2, τ_3, \dots (see also Foss and Zachary (2002)). The argument then proceeds in a two-sided version of that given above to obtain the final result more directly. As a proof of Veraverbeke’s Theorem this is somewhat tidier. However, the separate proof of the lower bound as given here (which, as previously remarked, is part of the folklore of the theorem) makes it clear that only the condition that F^s be long-tailed, rather than that it be subexponential, is required for that. Further, the proof of the lower bound requires only the Weak, rather than the Strong, Law of Large Numbers. Thus for given ϵ, δ , it is easy to determine a suitable constant L such that (7) holds, and so (8) may be used to give an explicit lower bound for $P(M > x)$ for any given x . (Of course this should be optimised over ϵ, δ .)

While the relation (16) in principle gives an upper bound for $P(M > x)$ for any given x (in which, for small γ , the importance of accurately bounding convolutions is greatly reduced), the more explicit determination of useful upper bounds remains a challenging problem. For further results on bounds in particular cases, including those of the Weibull and regularly varying distributions, see Kalashnikov and Tsitsiashvili (1999).

Finally, we remark also that a proof of the converse of Veraverbeke’s Theorem—that (for $\mathbf{E}\xi_1$ finite and negative) the relation (6) implies the subexponentiality of F^s —is given by Korshunov (1997).

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